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Influence of rotatory inertia and transverse shear on stochastic instability of the cross-ply laminated beam

Ratko Pavlović *, Ivan Pavlović

Mechanical Engineering Faculty, University of Niš, A. Medvedeva 14, 18000 Niš, Serbia and Montenegro

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Abstract

The stochastic instability problem associated with an axially loaded cross-ply laminated beam is formulated. The effects of shear deformation and rotatory inertia are included in the present formulations. The beam is subjected to time-dependent deterministic and stochastic forces. By using the direct Liapunov method, bounds for the almost sure instability of beams as a function of viscous damping coefficient, variance of the stochastic force, ratio of principal lamina stiffnesses, shear correction factor, number of layers, mode numbers and geometrical ratio, are obtained. Numerical calculations are performed for the Gaussian process with a zero mean and variance σ^2 as well as for harmonic process with an amplitude A.

Keywords: Stochastic instability; Liapunov functional; Shear correction factor; Rotatory inertia; Gaussian and harmonic process

1. Introduction

Composite materials are very suitable for structural applications where high strength-to-weight and stiffness-to-weight ratios are required. Their applications have increased considerably in the past few decades. Laminated composite materials are used as structural components in various applications (aerospace, automotive, marine, etc.). While these structural components can be plates or shells, beams are often encountered among these applications.

Deformation due to transverse shear strains plays a significant role in the behaviour of beams, plates and shells when a linear dimension or response mode wavelength is of the same order as the thickness.

* Corresponding author. Tel.: +381 18 500 666; fax: +381 18 524 930.
E-mail address: ratko@masfak.ni.ac.yu (R. Pavlović).

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For typical fibrous composite materials the transverse shear elastic modulus is less than 1/50 of the major in-surface normal modulus. Due to this high normal to shear modulus ratio in composite, the effect of transverse shear is more predominant than isotropic beam. Even in a composite beam with relatively smaller thickness or higher slenderness ratio, the effect of neglecting transverse shear may lead to a serious error.

The increasing use of fibrous composite structures has motivated many studies of anisotropic beams, plates and shells including the effects of rotatory inertia and transverse shear deformation.

Whitney and Pagano (1970) developed a bending theory for anisotropic laminated plates, including shear deformation and rotatory inertia. The governing equations reveal that unsymmetrically laminated plates display the same bending-extensional coupling phenomenon found in classical laminated plate theory based on the Kirchhoff assumptions. For certain fiber-reinforced composite materials, a radical departure from the classical laminated plate theory is indicated.

Krishnaswamy et al. (1992) used a series solution in conjuction with Lagrange multipliers to solve the free vibration problem of generally layered composite beams. The effect of shear deformation and rotatory inertia are included in the formulation.

In most of the papers, dissipation of energy was described by an external viscous model of damping. For such a model Kozin (1972) introduced the "best" Liapunov's functional, suitable for studying of almost sure asymptotic stability of beams and plates axially loaded by zero-mean stationary ergodic forces, whose samples are continuous with probability one.

Tylikowski (1992) considered stochastic stability of composite viscoelastic beams, where the influence of standard model parameters and eigenfrequencies on stability domains is shown.

The influence of rotatory inertia and shear deformations on the dynamic instability of isotropic, elastic beams subjected to random excitations is studied by Pavlović and Kozić (1993). Their major conclusion is that instability regions change qualitatively when the shear is taken into account, and rotatory inertia can be neglected.

The direct Liapunov method is used by Pavlović et al. (2001, 2004) for investigating the stochastic instability of elastic and viscoelastic beams including the effect of transverse shear, and sufficient conditions for almost sure asymptotic instability as a function of the damping coefficient or reduced retardation time and beam parameters are derived.

It is well known that the effect of transverse shear can be significant when the cross-sectional dimensions of a beam are large in comparison to its length; the effect of rotatory inertia for higher modes is important as well.

The purpose of the present paper is the investigation of dynamic instability of elastic, laminated beams subjected to time-dependent axial forces when transverse shear and rotatory inertia are taken into account. By using recent results of the stochastic processes theory we can apply the Liapunov method to obtain sufficient criteria for almost sure asymptotic instability in terms of the damping coefficient, variance of the stochastic force, ratio of principal lamina stiffnesses, shear correction factor, number of layers, mode number, geometric ratio and deterministic component of axial loading. The principal contribution of this paper is that the influence of transverse shear and rotatory inertia on almost sure instability regions can be more significant for various composite materials than for isotropic ones.

2. Problem formulation

Let us consider an elastic, composite beam of length ℓ, subjected to uniformly distributed stochastic axial time-dependent loading. The beam is made up of many unidirectional plies stacked up in 0° or 90° with respect to a reference axis. Such cross-ply beams can be symmetric or antisymmetric with $A = b \times h$ cross-sectional area, as shown in Fig. 1.

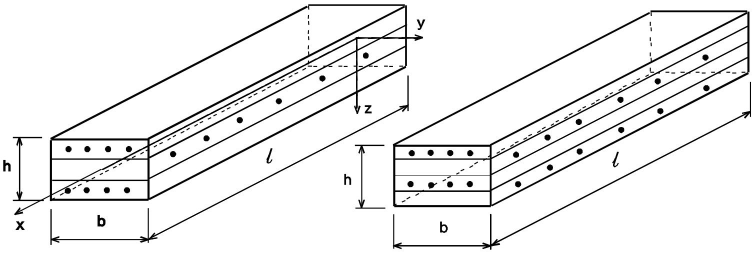

Fig. 1. Geometry of cross-ply symmetric and antisymmetric beams.

The dynamic equilibrium equations, based on the first order shear deformation theory, after Krishnaswamy et al. (1992), are

$$A_{11}\frac{\partial^2 u}{\partial X^2}+B_{11}\frac{\partial^2 \psi}{\partial X^2}=0, \tag{1}$$

$$\rho h\left(\frac{\partial^2 W}{\partial T^2}+2\beta_1\frac{\partial W}{\partial T}\right)+F(T)\frac{\partial^2 W}{\partial X^2}-kA_{55}\left(\frac{\partial \psi}{\partial X}+\frac{\partial^2 W}{\partial X^2}\right)=0, \tag{2}$$

$$\rho I\left(\frac{\partial^2 \psi}{\partial T^2}+2\beta_1\frac{\partial \psi}{\partial T}\right)-B_{11}\frac{\partial^2 u}{\partial X^2}-D_{11}\frac{\partial^2 \psi}{\partial X^2}+kA_{55}\left(\psi+\frac{\partial W}{\partial X}\right)=0, \tag{3}$$

where A_{11}, A_{55}, B_{11} and D_{11} are laminate stiffness coefficients, u and W are the mid-plane displacements in the X and Z directions, ψ is bending slope, k is shear correction factor, ρ is the mass density, $I = h^3/12$, cross-sectional moment of inertia, T is the time and $F(T)$ is stochastic axial force. By differential equations (1)–(3), Whitney and Pagano (1970) also describe dynamic behaviour of laminated composite plates in cylindrical bending. This system can be reduced to two equations:

$$\rho h\left(\frac{\partial^2 W}{\partial T^2}+\beta_1\frac{\partial W}{\partial T}\right)+F(T)\frac{\partial^2 W}{\partial X^2}-kA_{55}\left(\frac{\partial \psi}{\partial X}+\frac{\partial^2 W}{\partial X^2}\right)=0, \tag{4}$$

$$\rho I\left(\frac{\partial^2 \psi}{\partial T^2}+2\beta_1\frac{\partial \psi}{\partial T}\right)-\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)\frac{\partial^2 \psi}{\partial X^2}+kA_{55}\left(\psi+\frac{\partial W}{\partial X}\right)=0. \tag{5}$$

The following parameters can be used to non-dimensionalize equations (4) and (5):

$$X=\ell x,\quad W=\ell w,\quad T=k_t t,\quad k_t=\ell^2\sqrt{\frac{\rho h}{D_{11}-\frac{B_{11}^2}{A_{11}}}},\quad \beta=k_t\beta_1,\quad s^2=12\left(\frac{\ell}{h}\right)^2,$$

$$e^2=\frac{A_{55}\ell^2}{D_{11}-\frac{B_{11}^2}{A_{11}}},\quad f_0+f(t)=\frac{F(T)\ell^2}{D_{11}-\frac{B_{11}^2}{A_{11}}}. \tag{6}$$

Substituting relations (6) into Eqs. (4) and (5), gives

$$\frac{\partial^2 w}{\partial t^2}+2\beta\frac{\partial w}{\partial t}-ke^2\left(\frac{\partial \psi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+(f_0+f(t))\frac{\partial^2 w}{\partial x^2}=0, \tag{7}$$

$$\frac{\partial^2 \psi}{\partial t^2}+2\beta\frac{\partial \psi}{\partial t}+ke^2s^2\left(\psi+\frac{\partial w}{\partial x}\right)-s^2\frac{\partial^2 \psi}{\partial x^2}=0. \tag{8}$$

Boundary conditions corresponding to simply supported edges have the form:

$$\left.\begin{array}{l} x=0 \\ x=1 \end{array}\right\} \quad w=0, \quad \frac{\partial \psi}{\partial x}=0. \tag{9}$$

3. Instability analysis

With the purpose of applying the Liapunov method, we can construct the functional by means of the modified Parks-Pritchard method (1969). Thus, let us write Eqs. (7) and (8) in formal form $\mathfrak{L}\mathfrak{U}=0$, where $\mathfrak{U}=\operatorname{col}(w,\psi)$ is matrix column, and introduce linear operator:

$$\mathfrak{N}=\begin{bmatrix} 2s^2\frac{\partial}{\partial t}+2s^2\beta & 0 \\ 0 & 2\frac{\partial}{\partial t}+2\beta \end{bmatrix}, \tag{10}$$

which is a formal derivative of the operator $\mathfrak{L}$ with respect to $\partial/\partial t$. Integrating the scalar product $\mathfrak{L}\mathfrak{U}\mathfrak{N}\mathfrak{U}$ on rectangular $C=\Omega\times\Delta=[x:0\leqslant x\leqslant 1]\times[\tau:0\leqslant\tau\leqslant t]$, it is clear that

$$\int_0^t\int_0^1 \mathfrak{L}\mathfrak{U}\mathfrak{N}\mathfrak{U}\,\mathrm{d}x\,\mathrm{d}\tau=0. \tag{11}$$

After applying partial integration to Eq. (11), the sum of two integrals may be obtained. In the first, integration is only on the spatial domain and we choose it to be the Liapunov functional:

$$\mathbf{V}=\int_0^1\left\{s^2\left(\frac{\partial w}{\partial t}+\beta w\right)^2+s^2\beta^2w^2-s^2f_0\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial \psi}{\partial t}+\beta\psi\right)^2 +\beta^2\psi^2+ke^2s^2\left(\psi+\frac{\partial w}{\partial x}\right)^2+s^2\left(\frac{\partial \psi}{\partial x}\right)^2\right\}\mathrm{d}x. \tag{12}$$

Since it is evident that

$$\mathbf{V}|_0^t-\int_0^t\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\,\mathrm{d}t=0 \tag{13}$$

then the second integral in (13) is a time derivative of the functional (12) along Eqs. (7) and (8):

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}=-\int_0^1\left\{2\beta s^2\left(\frac{\partial w}{\partial t}\right)^2+2\beta\left(\frac{\partial \psi}{\partial t}\right)^2-2\beta s^2f_0\left(\frac{\partial w}{\partial x}\right)^2+2s^2f(t)\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial t}+\beta w\right) +2\beta ke^2s^2\left(\psi+\frac{\partial w}{\partial x}\right)^2+2\beta s^2\left(\frac{\partial \psi}{\partial x}\right)^2\right\}\mathrm{d}x. \tag{14}$$

Functional (12) is positive-definite, if

$$\int_0^1\left\{-s^2f_0\left(\frac{\partial w}{\partial x}\right)^2+ke^2s^2\left(\psi+\frac{\partial w}{\partial x}\right)^2+s^2\left(\frac{\partial \psi}{\partial x}\right)^2\right\}\mathrm{d}x\geqslant 0. \tag{15}$$

According to the well-known Steklov inequality it may be written

$$\int_0^1\left(\frac{\partial \psi}{\partial x}\right)^2\mathrm{d}x\geqslant\pi^2\int_0^1\psi^2\,\mathrm{d}x \tag{16}$$

and relation (15) is satisfied if

$$f_0 \leqslant ke^2\left(1-\frac{ke^2}{\pi^2}\right), \tag{17}$$

which represents a condition for static critical force.

In order to estimate the functional derivative, the expression (14) can be written in the form

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = -2\beta\mathbf{V} + 2\mathbf{U}, \tag{18}$$

where $\mathbf{U}$ is the functional:

$$\mathbf{U} = \int_0^1 \left\{s^2\left(2\beta^2 w - f(t)\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial w}{\partial t}+\beta w\right) + 2\beta^2\psi\left(\frac{\partial \psi}{\partial t}+\beta\psi\right)\right\}\mathrm{d}x. \tag{19}$$

Let us introduce the scalar function $\lambda(t)$ defined as the minimum overall w, ψ, $v=\partial w/\partial t$ and $\omega=\partial\psi/\partial t$ of the ratio $\mathbf{U}/\mathbf{V}$:

$$\lambda(t) = \min_{w,\psi,v,\omega}\frac{\mathbf{U}}{\mathbf{V}}. \tag{20}$$

As a minimum point is a particular case of the stationary point, we may write

$$\delta(\mathbf{U}-\lambda\mathbf{V}) = 0. \tag{21}$$

By using the related Euler's equations we obtain

$$\begin{aligned}
&2\lambda\left[\beta(v+2\beta w) + f_0\frac{\partial^2 w}{\partial x^2} - ke^2\left(\frac{\partial\psi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)\right] - 2\beta^2(v+2\beta w) + f(t)\left(\frac{\partial^2 v}{\partial x^2}+2\beta\frac{\partial^2 w}{\partial x^2}\right) = 0,\\
&2\beta^2 w - f(t)\frac{\partial^2 w}{\partial x^2} - 2\lambda(v+\beta w) = 0,\\
&\lambda\left[\beta(\omega+2\beta\psi) + ke^2 s^2\left(\psi+\frac{\partial w}{\partial x}\right) - s^2\frac{\partial^2\psi}{\partial x^2}\right] - \beta^2(\omega+2\beta\psi) = 0,\\
&\beta^2\psi - \lambda(\omega+\beta\psi) = 0.
\end{aligned} \tag{22}$$

According to the differential equations (7), (8) and boundary conditions (9) we may take eigenfunctions for displacement $w\ \sin\alpha_m x$, and bending slope $\psi\ \cos\alpha_m x$, where $\alpha_m = m\pi$, m-mode number. Hence, from (22)

$$\lambda(t) = \min_m \sqrt{\frac{\beta^4 A_m + F_m^2 B_m + \sqrt{(\beta^4 A_m - F_m^2 B_m)^2 + 4\beta^4 k^2 e^4 s^2 \alpha_m^2 F_m^2}}{2(A_m B_m - k^2 e^4 s^2 \alpha_m^2)}}, \tag{23}$$

where

$$A_m = \beta^2 - f_0\alpha_m^2 + ke^2\alpha_m^2, \quad B_m = \beta^2 + s^2\alpha_m^2 + ke^2 s^2, \quad F_m = \beta^2 + \frac{1}{2}f(t)\alpha_m^2. \tag{24}$$

Using the property of function λ we can estimate the time derivative of functional as follows:

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} \geqslant 2(-\beta+\lambda)\mathbf{V}.$$

By solving this differential inequality, we obtain the following estimation of the functional:

$$\mathbf{V}(t) \geqslant \mathbf{V}(0)\exp\left[2t\left(-\beta + \frac{1}{t}\int_0^t \lambda(\tau)\,\mathrm{d}\tau\right)\right]. \tag{25}$$

Therefore, it can be stated that the trivial solution of Eqs. (7) and (8) is almost surely asymptotically unstable if

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \lambda(\tau)\,\mathrm{d}\tau \geqslant \beta, \tag{26}$$

or, when the process $f(t)$ is ergodic and stationary

$$E\{\lambda(t)\} \geqslant \beta, \tag{27}$$

where E denotes the operator of mathematical expectation.

4. Numerical results and discussion

The expression (23) and inequality (26) or (27) give the possibility to obtain critical damping coefficient, guaranteeing an almost sure asymptotic instability as function of the statistic characteristics of the loading. The almost sure asymptotic instability region is defined as a set where the damping coefficient is smaller than its critical value.

For symmetric cross-ply laminated beams is

$$e^2 = 6\frac{F}{(F-1)P+1}\left(\frac{\ell}{h}\right)^2, \qquad P = \frac{1}{(1+M)^3} + \frac{M(N-3)[M(N-1)+2(N+1)]}{(N^2-1)(1+M)^3} \tag{28}$$

and antisymmetric cross-ply laminated beam is

$$e^2 = 4\frac{F}{\frac{1+F}{3} - \frac{(F-1)^2}{N^2(1+F)}}\left(\frac{\ell}{h}\right)^2, \tag{29}$$

where N is total number of layers, $F = E_2/E_1$ is the ratio of principal lamina stiffnesses, E_1 and E_2 are major and minor Young's modulus, and M is cross-ply ratio defined as the ratio of the total thickness of odd-numbered layers to total thickness of even-numbered layers:

$$M = \frac{\sum_{i=\mathrm{odd}} h_i}{\sum_{i=\mathrm{even}} h_i}. \tag{30}$$

In numerous papers the comparisons are made for regular laminates (composed of layers with equal thickness). The cross-ply ratio for regular symmetric laminates is $M = (N+1)/(N-1)$, and for regular antisymmetric $M = 1$.

Here is a comparison of the symmetric cross-ply laminated beam with the regular antisymmetric one, so that both have cross-ply ratio one. For the symmetric case odd numbered layers have thickness $h/(N+1)$, and even numbered layers have thickness $h/(N-1)$.

Taking into account the distributional properties of the process $f(t)$, Kozin (1972) obtained more precise results than those obtained via Schwarz inequality. Supposing that probability density function $p(f)$ is known for the process $f(t)$, the boundaries of the almost sure instability are calculated by using the corresponding Gauss–Christoffel quadratures.

For Gaussian process we take the parameters of Gauss–Hermite quadrature, and for harmonic process we set $f(t) = A\cos(\omega t + \theta)$, where A, ω are fixed amplitude and frequency, and θ is a uniformly distributed random phase on the interval $[0, 2\pi)$. In order to compare both processes the variance of harmonic process $\sigma^2 = A^2/2$ is used, and we take the Gauss–Chebyshev quadrature.

Numerical results for cross-ply symmetric laminated beams, when the deterministic component of axial loading is zero, $f_0 = 0$ (that case is analysed earlier by Pavlovi et al. 2001 and 2004), are shown in Figs. 2–6.

The boundaries of the almost sure instability are presented with a full line for the Gaussian process, and dashed line for the harmonic process.

In Fig. 2 instability regions as a function of the ratio of principal lamina stiffnesses, $F = E_2/E_1$, for graphite-epoxy ($F = 1/40$), boron-epoxy ($F = 1/10$) and isotropic material ($F = 1$) are plotted. Instability regions increase with the decrease of ratio F, and they are lowermost for isotropic beams.

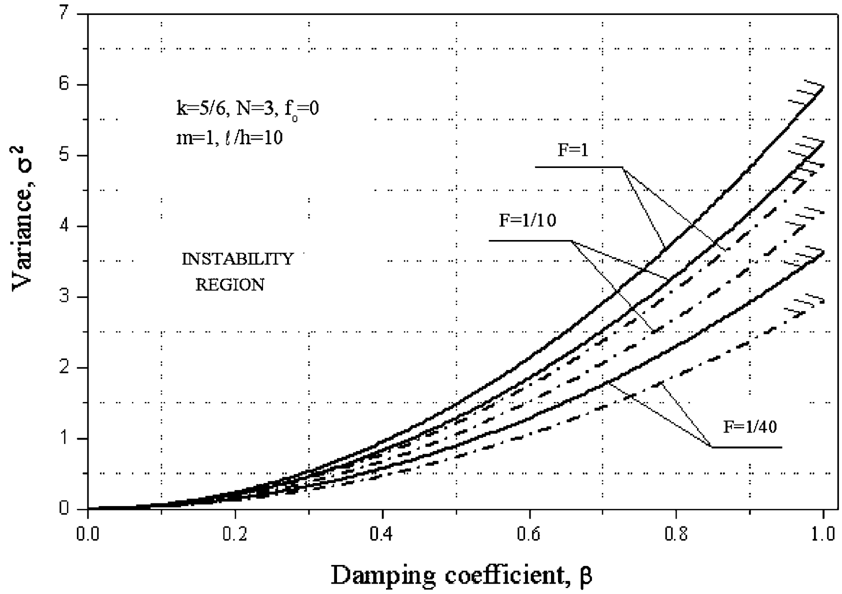

Fig. 2. Influence of the ratio of principal lamina stiffnesses on instability regions symmetric cross-ply laminated beams.

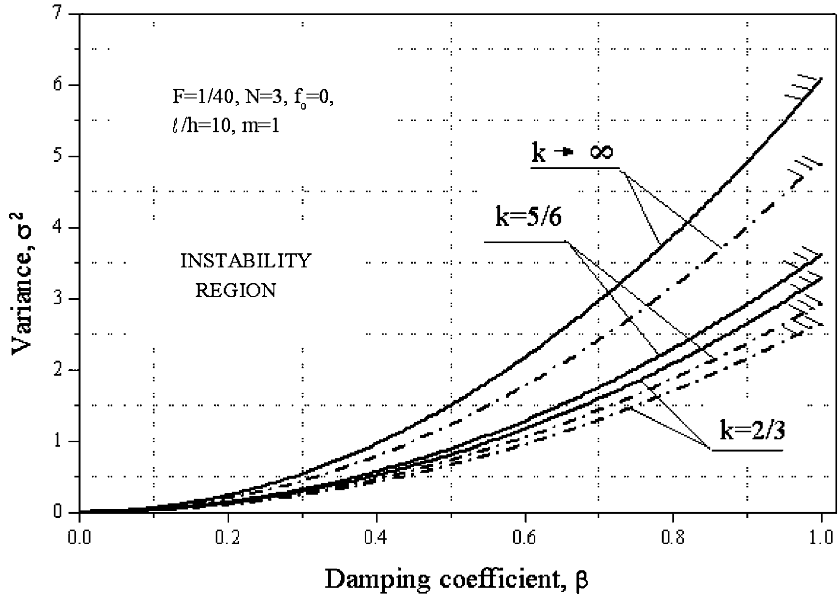

Fig. 3. Influence of the shear correction factor on instability regions symmetric cross-ply laminated beams.

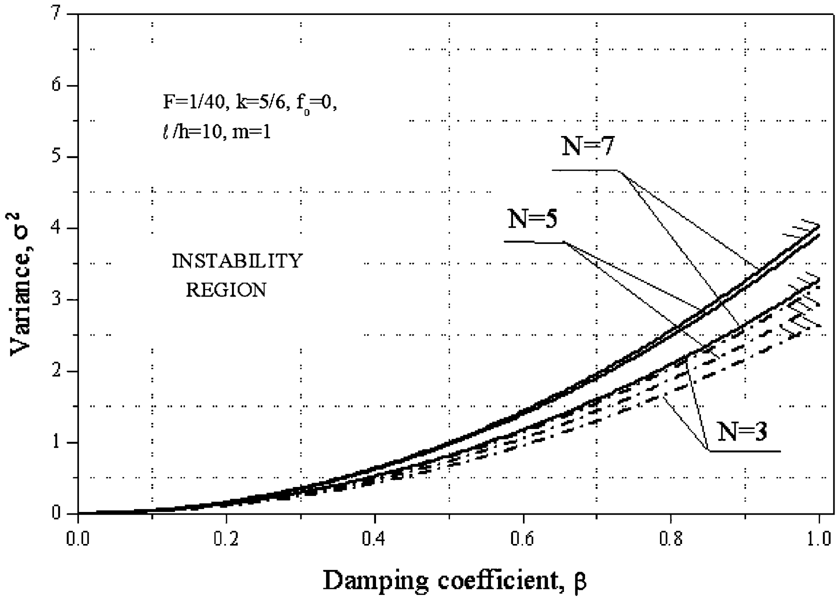

Fig. 4. Influence of the number of layers on instability regions symmetric cross-ply laminated beams.

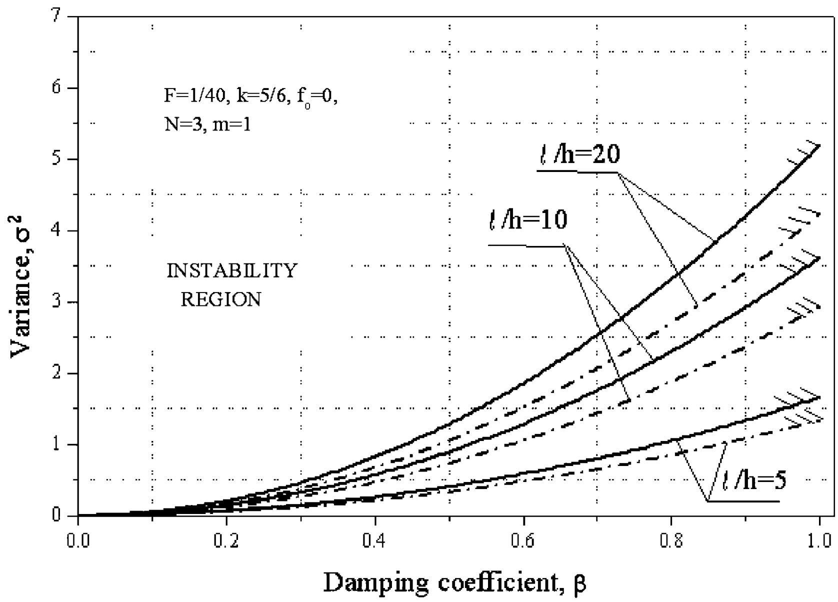

Fig. 5. Influence of the geometric ratio ℓ/h on instability regions symmetric cross-ply laminated beams.

By studying shear deformation in heterogeneous anisotropic plates, Whitney and Pagano (1970) conclude that shear correction factor $k = 5/6$ gives good correlation with the exact theory, while $k = 2/3$ is a better estimate for the symmetric cross-ply three-layered strip.

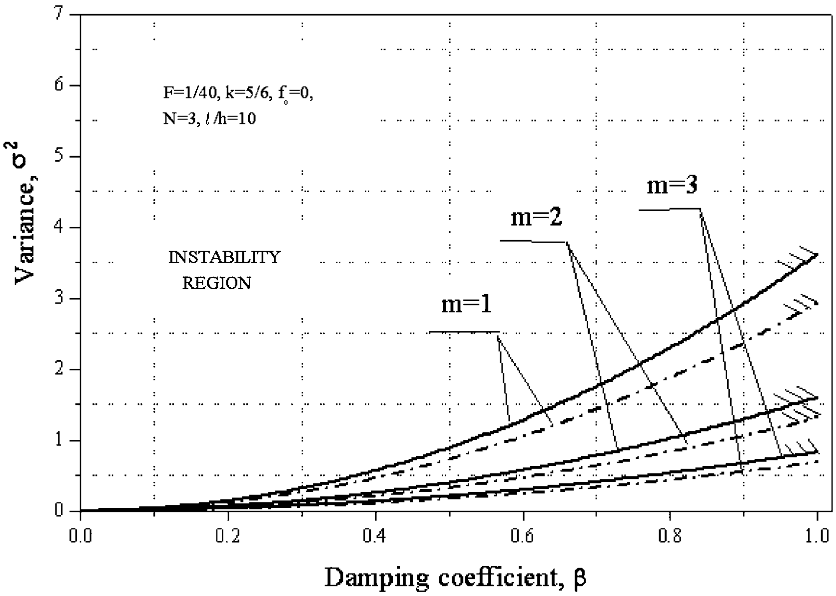

Fig. 6. Influence of the mode numbers on instability regions symmetric cross-ply laminated beams.

In Fig. 3 instability regions as a function of shear correction factor are given. Numerical calculation is performed for the cases when the effect of transverse shear is neglected ($k \to \infty$), $k = 5/6$ and $k = 2/3$. It is evident that neglecting the transverse shear will be an error. Generally speaking, the influence of transverse shear leads to the growth of almost sure regions of instability.

In Fig. 4 instability regions as a function of the number of layers N, are shown. Increase of the number of layers brings about decrease of instability regions, so that the smallest region of instability will occur in the case when $N \to \infty$, i.e. in the case of the isotropic beam.

The recent results (Banerjee and Williams, 1994) showed that the effect of shear deformation diminishes with increase in the slenderness ratio ℓ/h of the beam. By direct observation from Fig. 5, it is evident that increase in slenderness ratio, leads to decrease of the instability region.

When a continuous system is subjected to time-dependent stochastic load, it is very important to find stability or instability regions as a function of mode numbers.

By studying the stability of a moving elastic strip subjected to random parametric excitation, Kozin and Milstead (1979) showed almost sure asymptotic stability regions for the first mode, as a function of mode number and for the infinite mode.

Similarly as in the case of isotropic beams (Pavlović et al., 2004), from Fig. 6 it may be seen that instability regions are larger on higher mode numbers, ($m = 2, 3$).

Numerical results for cross-ply antisymmetric laminated beams, also without the deterministic component of axial loading as a function of the ratio of principal lamina stiffnesses, shear correction factor, number of layers, geometrical ratio ℓ/h and mode numbers are shown in Figs. 7–11.

Except for the number of layers, the influence of transverse shear on instability regions is the same as for cross-ply symmetric laminated beams. In the case of antisymmetric cross-ply laminated beams, increase in the number of layers leads to the increase of instability regions, Fig. 9. If the number of layers tends to infinity ($N \to \infty$), then expressions (28) and (29) are equal:

$$e^2 = 12\frac{F}{F+1}\left(\frac{\ell}{h}\right)^2 \tag{31}$$

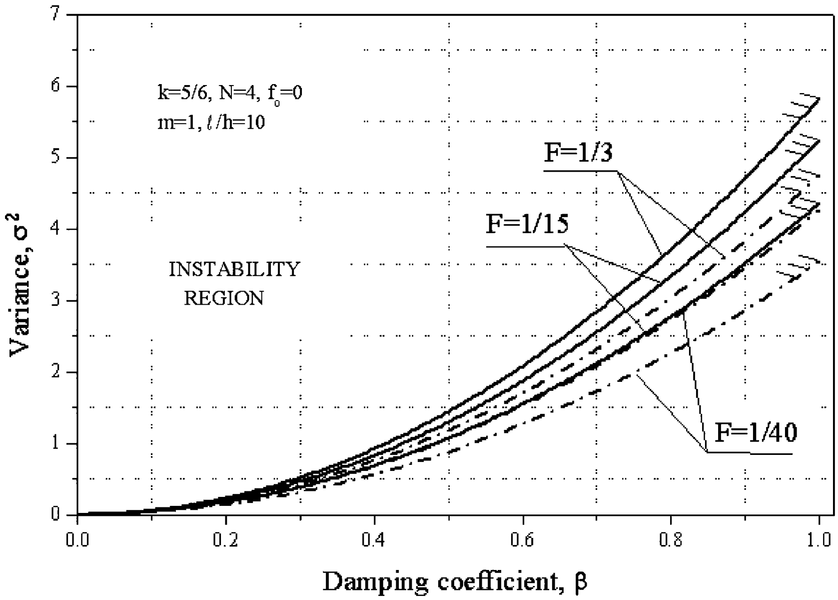

Fig. 7. Influence of the ratio of principal lamina stiffnesses on instability regions antisymmetric cross-ply laminated beams.

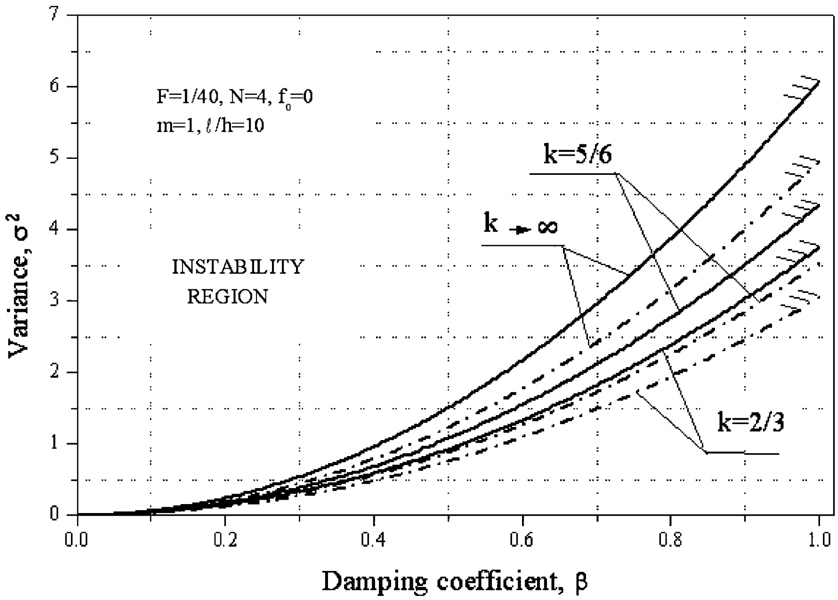

Fig. 8. Influence of the shear correction factor on instability regions antisymmetric cross-ply laminated beams.

and we have a unique limit curve for both types of laminated beams. In the case of a finite number of layers for symmetric cross-ply beams the curves are right, and for antisymmetric ones left from the limit curve. However, for seven and eight layered beams, the difference between curves is less than 3%.

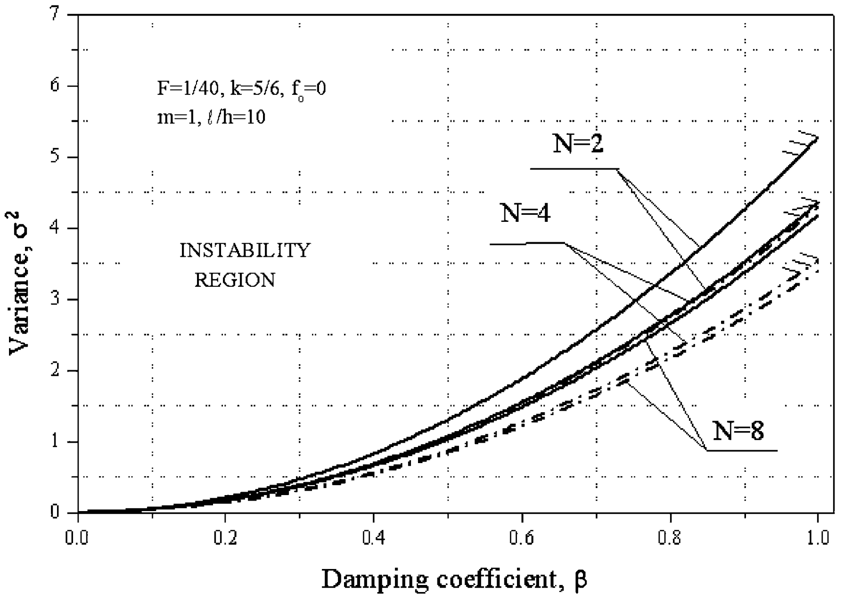

Fig. 9. Influence of the number of layers on instability regions antisymmetric cross-ply laminated beams.

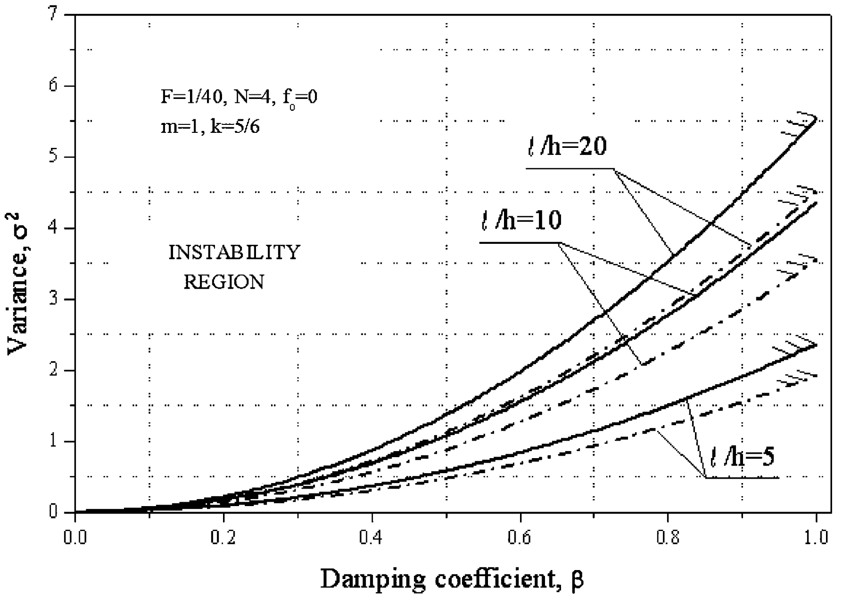

Fig. 10. Influence of the geometric ratio ℓ/h on instability regions antisymmetric cross-ply laminated beams.

Numerical results for cross-ply laminates with $M = 1$ show that regular antisymmetric have some less instability regions than symmetric. The use of regular laminates can be recommended, not only for economic reasons, but also on the stability stage.

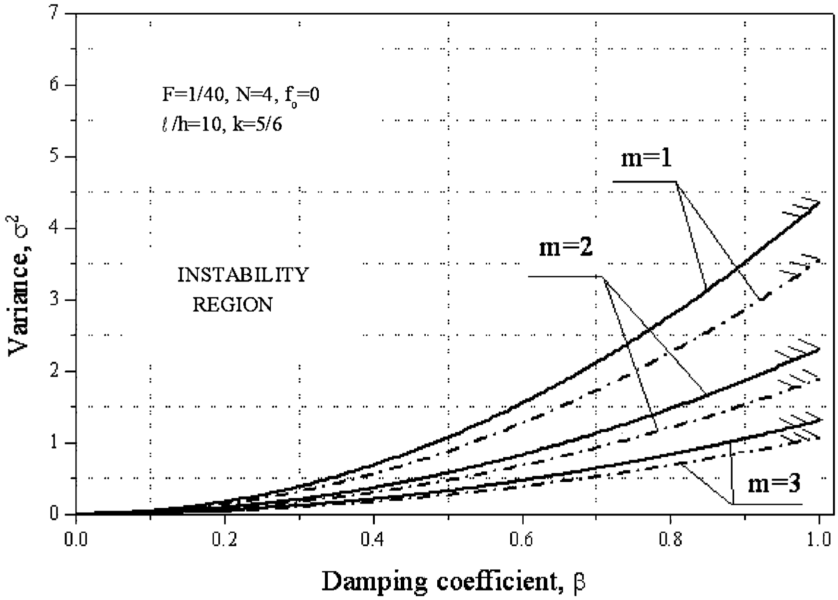

Fig. 11. Influence of the mode numbers on instability regions antisymmetric cross-ply laminated beams.

As for the influence of cross-section rotatory inertia on the almost sure stochastic instability regions of symmetric cross-ply laminated beams, all numerical calculations, even for higher mode numbers, indicate that we can neglect this influence. It is evident from Table 1 that absolute error is very small.

For the antisymmetric cross-ply beam, rotatory inertia is more significant, and cannot be ignored. From Table 2 it is clear that, by neglecting rotatory inertia in the first mode ($m = 1$), we made an error of about 5%, and for the second mode ($m = 2$), the error is even greater than 11%. These conclusions are valid for both processes.

If we omit rotatory inertia at the beginning and restrict only to analysing the influence of transverse shear on stochastic instability of composite beams, then the analytical solution is simpler, and it is given in the Appendix A.

Table 1
Loading variance for symmetric cross-ply beam where is rotatory inertia neglected (I) and included (II) $F = 1/40$, $N = 3$, $\ell/h = 10$, $k = 5/6$

	β					
		0.2	0.4	0.6	0.8	1.0
Gaussian process						
First mode ($m = 1$)	I	0.14249	0.57116	1.28958	2.30367	3.62176
	II	0.14249	0.57116	1.28956	2.30362	3.62164
Second mode ($m = 2$)	I	0.063908	0.255705	0.575614	1.02401	1.6014
	II	0.063908	0.255705	0.575614	1.02401	1.6014
Harmonic process						
First mode ($m = 1$)	I	0.11569	0.46733	1.05333	1.87673	2.94289
	II	0.11569	0.46732	1.05329	1.87666	2.94272
Second mode ($m = 2$)	I	0.05229	0.209222	0.470977	0.837859	1.31029
	II	0.05229	0.209221	0.470975	0.837854	1.31027

Table 2
Loading variance for antisymmetric cross-ply beam where is rotatory inertia neglected (I) and included (II) $F = 1/40$, $N = 4$, $\ell/h = 10$, $k = 5/6$

		β				
		0.2	0.4	0.6	0.8	1.0
Gaussian process						
First mode ($m = 1$)	I	0.18058	0.72351	1.63235	2.91304	4.57391
	II	0.17194	0.68893	1.55454	2.77468	4.35766
Second mode ($m = 2$)	I	0.102874	0.41157	0.926371	1.64747	2.57556
	II	0.092291	0.369238	0.831063	1.47814	2.31098
Harmonic process						
First mode ($m = 1$)	I	0.14775	0.59199	1. 33448	2.37594	3.72195
	II	0.14068	0.56368	1.27058	2.26251	3.54486
Second mode ($m = 2$)	I	0.084173	0.336753	0.757922	1.34798	2.10736
	II	0.075514	0.302116	0.679988	1.20943	1.89086

5. Conclusion

This paper discusses dynamic instability of cross-ply laminated beams when transverse shear and rotatory inertia are taken into account.

As in the isotropic beam, in the case of the cross-ply symmetric and antisymmetric laminated beam, where cross-sectional dimensions are large in comparison to length, transverse shear can be significant and cannot be neglected.

Rotatory inertia can be neglected in discussing the symmetric cross-ply laminated beam, but it may be a serious error in the case of the antisymmetric cross-ply beam.

Function $\lambda(t)$ given by (23) can be used for both cross-ply beams, but in the symmetric case we can use $\lambda(t)$ given by relation (*f*) in Appendix A.

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Appendix A

If we neglect cross-section rotatory inertia, in Eqs. (1)–(3), (omitting first two terms in Eq. (8)) the instability analysis is similar as in the case isotropic beams, Pavlović et al. (2004), Eqs. (1)–(3) reduce on only one equation:

$$\left(1-\frac{1}{ke^2}\frac{\partial^2}{\partial x^2}\right)\left[\frac{\partial^2 w}{\partial t^2}+2\beta\frac{\partial w}{\partial t}+(f_0+f(\omega,t))\frac{\partial^2 w}{\partial x^2}\right]+\frac{\partial^4 w}{\partial x^4}=0. \tag{a}$$

Liapunov functional has the form

$$\mathbf{V} = \int_0^1 \left\{ \left(\frac{\partial w}{\partial t} + \beta w - \frac{1}{ke^2} \frac{\partial^3 w}{\partial^2 x \, \partial t} - \frac{\beta}{ke^2} \frac{\partial^2 w}{\partial x^2} \right)^2 + \beta^2 \left[w^2 + \frac{2}{ke^2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{k^2 e^4} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] + \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{1}{ke^2} \left(\frac{\partial^3 w}{\partial x^3} \right)^2 - f_0 \left[\left(\frac{\partial w}{\partial x} \right)^2 + \frac{2}{ke^2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{1}{k^2 e^4} \left(\frac{\partial^3 w}{\partial x^3} \right)^2 \right] \right\} \mathrm{d}z. \tag{b}$$

And time derivative is

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = -\int_0^1 \left\{ 2\beta \left[\left(\frac{\partial w}{\partial t} \right)^2 + \frac{2}{ke^2} \left(\frac{\partial^2 w}{\partial x \, \partial t} \right)^2 + \frac{1}{k^2 e^4} \left(\frac{\partial^3 w}{\partial x^2 \, \partial t} \right)^2 \right] - 2\beta f_0 \left[\left(\frac{\partial w}{\partial x} \right)^2 + \frac{2}{ke^2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{1}{k^2 e^4} \left(\frac{\partial^3 w}{\partial x^3} \right) \right] + 2\beta \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{1}{ke^2} \left(\frac{\partial^3 w}{\partial x^3} \right)^2 \right] + 2f(t) \left(\frac{\partial^2 w}{\partial x^2} - \frac{1}{ke^2} \frac{\partial^4 w}{\partial x^4} \right) \left[\left(\frac{\partial w}{\partial t} + \beta w - \frac{1}{ke^2} \frac{\partial^3 w}{\partial x^2 \, \partial t} - \frac{\beta}{ke^2} \frac{\partial^2 w}{\partial x^2} \right)^2 \right] \right\} \mathrm{d}z. \tag{c}$$

Scalar function $\lambda(t)$ can be determined from

$$\delta(\dot{V} - \lambda V) = 0. \tag{d}$$

Hence, final expression for unknown function $\lambda(t)$ is

$$\lambda(t) = \min_m \lambda_m(t), \tag{e}$$

where

$$\lambda_m = -2\beta + \frac{|2\beta^2 + f(t)\alpha_m^2|}{\sqrt{\beta^2 + \alpha_m^2 \left(\frac{\alpha_m^2}{1 + \alpha_m^2/(ke^2)} - f_0 \right)}}. \tag{f}$$

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